A q-LINEAR ANALOGUE OF THE PLANE WAVE EXPANSION

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Abstract. We obtain a q-linear analogue of Gegenbauer's expansion of the plane wave. It is expanded in terms of the little q-Gegenbauer polynomials and the *third* Jackson q-Bessel function. The result is obtained by using a method based on bilinear biorthogonal expansions.

1. Introduction

Let $\beta > -1/2$. Gegenbauer's expansion of the plane wave in Gegenbauer polynomials and Bessel functions is

$$e^{ixt} = \Gamma(\beta)\left(\frac{x}{2}\right)^{-\beta}\sum_{n=0}^{\infty} i^n(\beta+n)J_{\beta+n}(x)C_n^{\beta}(t), \quad t\in[-1,1] \tag{1}$$

(see [21, Ch. 11, § 5, formula (2)]). In [12, formula (3.32)], Ismail and Zhang have discovered a basic analogue of (1) on q-quadratic grids, expanding the q-quadratic exponential function (a solution of a first order equation involving the so-called Askey-Wilson operator) in terms of *second* Jackson q-Bessel functions, $J_\nu^{(2)}(z;q)$, and the continuous q-Gegenbauer polynomials (moreover, this has been later extended to continuous q-Jacobi polynomials in [10]). Their q-quadratic exponential inherits the orthogonality of the continuous q-Gegenbauer polynomials, and leads to a theory of Fourier series on q-quadratic grids (see [4] and [20]). Since then, it has been a folk open question to find a q-analogue of (1) involving discrete q-Gegenbauer polynomials and q-Bessel functions of a different type. In this note, and also for $\beta > -1/2$, we will obtain the following q-analogue of (1):

$$e(itx;q^2) = \frac{(q^2;q^2)_\infty}{(q^{2\beta};q^2)_\infty}\,x^{-\beta}\sum_{n=0}^{\infty} i^n q^{-[\frac{n+1}{2}](\beta-\frac{1}{2})}(1-q^{2\beta+2n})J^{(3)}_{\beta+n}(xq^{[\frac{n+1}{2}]};q^2)C_n^{\beta}(t;q^2), \tag{2}$$

for $t\in[-1,1]$. Here, $[r]$ denotes the biggest integer less or equal than r, $J_\nu^{(3)}(z;q)$ is the *third* Jackson q-Bessel function, and $C_n^{\beta}(t;q^2)$ are q-analogues of the Gegenbauer polynomials, defined in terms of the little q-Jacobi polynomials (see definitions of all these functions in the third section of the paper). In fact we prove a more general formula than (2), see Theorem 2.

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The q-exponential function in (2) is the one introduced in [19]:

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2),$$

where

$$\cos(z;q^2) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{\frac{1}{2}} J^{(3)}_{-\frac{1}{2}}(z;q^2) \quad \text{and} \quad \sin(z;q^2) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{\frac{1}{2}} J^{(3)}_{\frac{1}{2}}(z;q^2).$$

The expansion (2) is obtained as a special case of a more general formula, which is a q-analogue of the expansion of the Dunkl kernel in terms of Bessel functions and generalized Gegenbauer polynomials (see [2] and [18]). The technique of proof is based on the method of Bilinear Biorthogonal Expansions, developed in [2] and provides as a byproduct, q-analogues of Neumann series which are valid for functions which belong to certain q-analogues of the Paley-Wiener space. We remark that another q-linear analogue (but involving completely different functions) of (1) has been obtained in [11].

The paper is organized as follows. We describe the setup of the method of Bilinear Biorthogonal Expansions in the next section. In the third section we collect some material on basic hypergeometric functions and q-integration and apply it in the fourth section to the context of the general set-up, yielding our main result. The last section contains the evaluation of some q-integrals which are essential in the proofs.

2. The method of Bilinear Biorthogonal Expansions

We proceed to describe the set-up of the method of Bilinear Biorthogonal Expansions [2]. The method aims to finding a bilinear expansion for $K(x,t)$, a function of two variables defined on $\Omega \times \Omega \subset \mathbb{R} \times \mathbb{R}$ and such that $K(x,t) = K(t,x)$ almost everywhere for $(x,t) \in \Omega \times \Omega$. It consists of three ingredients:

(i) First define on $L^2(\Omega, d\mu)$, with $d\mu$ a non-negative real measure, an integral transformation $\mathcal{K}$ with inverse $\widetilde{\mathcal{K}}$,

$$(\mathcal{K}f)(t) = \int_\Omega f(x)\overline{K(x,t)}\,d\mu(x), \qquad (\widetilde{\mathcal{K}}g)(x) = \int_\Omega g(t)K(x,t)\,d\mu(t).$$

As usual, it is enough to suppose that the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are defined with these formulas on a suitable dense subset of $L^2(\Omega, d\mu)$, and later extended to the whole $L^2(\Omega, d\mu)$ in the standard way. Let also note that, by Fubini's theorem, they satisfy

$$\int_\Omega (\mathcal{K}f)g\,d\mu = \int_\Omega (\mathcal{K}g)f\,d\mu, \qquad \int_\Omega (\widetilde{\mathcal{K}}f)g\,d\mu = \int_\Omega (\widetilde{\mathcal{K}}g)f\,d\mu.$$

(ii) Let $I \subset \Omega$ be an interval such that, as a function of t, $K(x,\cdot) \in L^2(I, d\mu)$ and consider the subspace $\mathcal{P}$ of $L^2(\Omega, d\mu)$ constituted by those functions f such that $\mathcal{K}f$ vanishes outside of I. This can also be written as

$$\mathcal{P} = \Big\{ f \in L^2(\Omega) : f(x) = \int_I u(t)K(x,t)\,d\mu(t),\ u \in L^2(I, d\mu) \Big\}.$$

(iii) Finally, consider a pair of complete biorthonormal sequences of functions in $L^2(I, d\mu)$, $\{P_n\}_{n\in N}$ and $\{Q_n\}_{n\in N}$ (with N a subset of $\mathbb{Z}$) and define, in $L^2(\Omega, d\mu)$, the sequences of functions $\{S_n\}_{n\in N}$ and $\{T_n\}_{n\in N}$ given by

$$S_n(x) = \widetilde{\mathcal{K}}(\chi_I \overline{Q_n})(x), \quad x \in \Omega, \qquad T_n(x) = \overline{\mathcal{K}(\chi_I P_n)(x)}, \quad x \in \Omega$$

(note that if $P_n = Q_n$ then $S_n = T_n$).

Then, the following holds (see [2, Theorem 1]):

Theorem 1. *For each $x \in \Omega$, the following expansion* [1] *holds in $L^2(I, d\mu)$:*

$$K(x,t) = \sum_{n \in N} P_n(t) S_n(x), \quad t \in I. \tag{3}$$

Moreover, $\{S_n\}_{n\in N}$ and $\{T_n\}_{n\in N}$ are a pair of complete biorthogonal sequences in $\mathcal{P}$, in such a way that every $f \in \mathcal{P}$ can be written as

$$f(x) = \sum_{n \in N} c_n(f) S_n(x), \quad x \in \Omega,$$

with

$$c_n(f) = \int_\Omega f(t)\overline{T_n(t)}\, d\mu(t).$$

The convergence is uniform in every set where $\|K(x,\cdot)\|_{L^2(I,d\mu)}$ is bounded.

3. Preliminaries on q-special functions

We follow the standard notations (see [8] and [14]). Choose a number q such that $0 < q < 1$. The notational conventions

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n, \quad (a_1, \dots, a_m; q)_n = \prod_{l=1}^{m} (a_l;q)_n, \quad |q| < 1,$$

where $n = 1, 2, \dots$ will be used. The symbol ${}_s\phi_r$ stands for the function

$${}_s\phi_r \left(\begin{matrix} a_1, \dots, a_s \\ b_1, \dots, b_r \end{matrix} \middle| q; z \right) = \sum_{n=0}^{\infty} \frac{\left((-1)^n q^{n(n-1)/2}\right)^{r-s+1} (a_1, \dots, a_s; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n;$$

in particular,

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} \middle| q; z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n.$$

We will also require the definition of the q-integral. The q-integral in the interval $(0, a]$ is defined as

$$\int_0^a f(t)\, d_q t = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n$$

and in the interval $(0, \infty)$ as

$$\int_0^\infty f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,$$

provided that the infinite sums converge absolutely. This can be extended to the whole real line in an obvious way.

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[1] The condition $t \in I$ in the identity (3) is *not* a mistake. Although $K(x,t)$ is defined on $\Omega \times \Omega$, the functions $P_n(t)$ are defined, in general, only on I.

The *third Jackson q-Bessel function* $J_\nu^{(3)}$ is defined by the power series

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n (q;q)_n}\, x^{2n}$$
$$= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, x^\nu\, {}_1\phi_1\left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}\,\middle|\, q; qx^2\right).$$

Throughout this paper, when no confusion is possible, we will drop the superscript and write simply

$$J_\nu(x;q) = J_\nu^{(3)}(x;q).$$

For $x \in (0,1)$, the *little q-Jacobi* polynomials are defined for $\alpha, \beta > -1$ by

$$p_n(x; q^\alpha, q^\beta; q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1} \\ q^{\alpha+1} \end{matrix}\,\middle|\, q; qx\right).$$

They satisfy the following discrete orthogonality relation (see [14, (14.12.2)]):

$$\int_0^1 \frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\, x^\alpha p_n(x;q^\alpha,q^\beta;q) p_m(x;q^\alpha,q^\beta;q)\, d_q x$$
$$= \frac{(1-q)(1-q^{\alpha+\beta+1})}{1-q^{\alpha+\beta+2n+1}} \frac{(q,q^{\alpha+\beta+2};q)_\infty}{(q^{\alpha+1},q^{\beta+1};q)_\infty} \frac{(q,q^{\beta+1};q)_n}{(q^{\alpha+1},q^{\alpha+\beta+1};q)_n}\, q^{n(\alpha+1)} \delta_{m,n}.$$

For our purposes we need to rewrite this orthogonality. We will use the polynomials $p_n^{(\alpha,\beta)}$ normalized as follows:

$$p_n^{(\alpha,\beta)}(x;q) = q^{-\frac{n(\alpha+1)}{2}} \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\, p_n(x;q^\alpha,q^\beta;q). \tag{4}$$

These polynomials satisfy

$$\lim_{q\to 1} p_n^{(\alpha,\beta)}(x;q) = P_n^{(\alpha,\beta)}(1-2x),$$

where $P_n^{(\alpha,\beta)}$ are the classical Jacobi polynomials (see [9, p. 478]). It will be convenient to replace q by q^2 in the above orthogonality. Then, from the definition of the q-integral we obtain the identity

$$\int_0^1 f(x)\, d_{q^2} x = (1+q) \int_0^1 x f(x^2)\, d_q x,$$

and use it in order to obtain the following:

$$\int_0^1 \frac{(q^2x^2;q^2)_\infty}{(q^{2\beta+2}x^2;q^2)_\infty} p_n^{(\alpha,\beta)}(x^2;q^2) p_m^{(\alpha,\beta)}(x^2;q^2) x^{2\alpha+1}\, d_q x$$
$$= \frac{1-q}{1-q^{2\alpha+2\beta+4n+2}} \frac{(q^{2n+2},q^{2\alpha+2\beta+2n+2};q^2)_\infty}{(q^{2\alpha+2n+2},q^{2\beta+2n+2};q^2)_\infty}\, \delta_{m,n}. \tag{5}$$

4. Application of the method

4.1. **The integral transform.** We will construct the integral transform required in the first ingredient of our method. A generalized q-exponential kernel (in the

spirit of the kernel for the Dunkl transform) can be defined in terms of q-Bessel. Indeed, we can consider the function

$$E_\alpha(ix;q^2) = \frac{(q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}\left(\frac{J_\alpha(x;q^2)}{x^\alpha} + \frac{J_{\alpha+1}(x;q^2)}{x^{\alpha+1}}\,xi\right) \tag{6}$$
$$= {}_1\phi_1\left(\left.\begin{matrix}0\\ q^{2\alpha+2}\end{matrix}\,\right|\,q^2;q^2x^2\right) + \frac{ix}{1-q^{2\alpha+2}}\,{}_1\phi_1\left(\left.\begin{matrix}0\\ q^{2\alpha+4}\end{matrix}\,\right|\,q^2;q^2x^2\right).$$

Taking the measure

$$d\mu_{q,\alpha}(x) = \frac{1}{2(1-q)}\frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}|x|^{2\alpha+1}\,d_qx,$$

in a similar way to the Dunkl transform, first introduced by Dunkl in [6] (see also [13] or [5]), for $\alpha \geq -1/2$ we can define the following q-integral transform:

$$\mathcal{F}_{\alpha,q}f(y) = \int_{-\infty}^{\infty} f(x)E_\alpha(-iyx;q^2)\,d\mu_{q,\alpha}(x), \quad y \in \{\pm q^k\}_{k\in\mathbb{Z}}, \tag{7}$$

for $f \in L^1(\mathbb{R}, d\mu_{q,\alpha})$. This q-integral transform is related to the q-Dunkl type operator introduced in [3] (for a different q-Dunkl type operator see [7]). The case $\alpha = -\frac{1}{2}$ provides a q-analogue of the Fourier transformation. In this special case, an inversion theory of this transform has been derived in [19] using the results of [16, 17]. For even functions this becomes a q-analogue of the Hankel transform

$$H_{\alpha,q}f(x) = \int_0^\infty \frac{J_\alpha(xy;q^2)}{(xy)^\alpha}\,f(y)\,d\omega_{q,\alpha}(y), \quad x > 0,$$

where $d\omega_{q,\alpha}(y) = \frac{y^{2\alpha+1}}{1-q}\,d_qy$. This is the transform studied by Koornwinder and Swarttouw [16, 17], up to a small modification. By the results in [16, 17] we have the inversion formula

$$f(q^n) = H_{\alpha,q}(H_{\alpha,q}f)(q^n).$$

In particular, $H_{\alpha,q}$ is an isometric transformation in $L^2((0,\infty), d\omega_{q,\alpha})$. For odd functions, $\mathcal{F}_{\alpha,q}$ turns down to $H_{\alpha+1,q}$.

For $\mathcal{F}_{\alpha,q}$, by combining the results for odd and even functions (or using again the arguments in [16, 17]), it is easy to check that $\mathcal{F}_{\alpha,q}^{-1}f(y) = \mathcal{F}_{\alpha,q}f(-y)$. Moreover, we have the formula

$$\int_{-\infty}^{\infty} u(y)\mathcal{F}_{\alpha,q}v(y)\,d\mu_{q,\alpha}(y) = \int_{-\infty}^{\infty} \mathcal{F}_{\alpha,q}u(y)v(y)\,d\mu_{q,\alpha}(y),$$

and $\mathcal{F}_{\alpha,q}$ is an isometry on $L^2(\mathbb{R}, d\mu_{q,\alpha})$. As in the case of the Dunkl transform, we can consider the parameter $\alpha > -1$.

4.2. **The space $\mathcal{P}$.** The space $\mathcal{P}$ is the following q-analogue of the Paley-Wiener space:

$$PW_{\alpha,q} = \left\{f \in L^2(\mathbb{R}, d\mu_{q,\alpha}) : f(t) = \int_{-1}^{1} u(x)E_\alpha(ixt;q^2)\,d\mu_{q,\alpha}(x),\ u \in L^2(\mathbb{R}, d\mu_{q,\alpha})\right\}.$$

4.3. **The biorthogonal functions.** Let us start defining the generalized little q-Gegenbauer polynomials and, later, we will take the corresponding q-Fourier-Neumann type series.

4.3.1. *Generalized little q-Gegenbauer polynomials.* To construct the plane wave expansion for the kernel (6), q-analogue of the Dunkl transform, we consider generalized little q-Gegenbauer polynomials

$$C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2) = (-1)^n \frac{(q^{2\alpha+2\beta+2};q^2)_n}{(q^{2\alpha+2};q^2)_n} p_n^{(\alpha,\beta)}(t^2;q^2),$$
$$C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t;q^2) = (-1)^n \frac{(q^{2\alpha+2\beta+2};q^2)_{n+1}}{(q^{2\alpha+2};q^2)_{n+1}} t p_n^{(\alpha+1,\beta)}(t^2;q^2),$$

where the polynomials $p_n^{(\alpha,\beta)}$ are defined by (4) in terms of the little q-Jacobi polynomials. Using (5) we obtain

$$\int_{-1}^{1} C_k^{(\beta+1/2,\alpha+1/2)}(t;q^2) C_j^{(\beta+1/2,\alpha+1/2)}(t;q^2) \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}\, d\mu_{q,\alpha}(t) = h_{k,q}^{(\beta,\alpha)} \delta_{k,j},$$

where

$$\begin{aligned} h_{2n,q}^{(\beta,\alpha)} &= \int_{-1}^{1} \Big[C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\Big]^2 \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}\, d\mu_{q,\alpha}(t) \\ &= \frac{1}{1-q^{2\alpha+2\beta+4n+2}} \frac{(q^{2\alpha+2\beta+2};q^2)_n}{(q^{2\alpha+2};q^2)_n} \frac{(q^{2n+2},q^{2\alpha+2\beta+2};q^2)_\infty}{(q^{2\beta+2n+2},q^2;q^2)_\infty}, \end{aligned}$$

$$\begin{aligned} h_{2n+1,q}^{(\beta,\alpha)} &= \int_{-1}^{1} \Big[C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\Big]^2 \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}\, d\mu_{q,\alpha}(t) \\ &= \frac{1}{1-q^{2\alpha+2\beta+4n+4}} \frac{(q^{2\alpha+2\beta+2};q^2)_{n+1}}{(q^{2\alpha+2};q^2)_{n+1}} \frac{(q^{2n+2},q^{2\alpha+2\beta+2};q^2)_\infty}{(q^{2\beta+2n+2},q^2;q^2)_\infty}. \end{aligned}$$

We will also consider the little q-Gegenbauer polynomials defined as

$$C_n^{\beta}(t;q^2) = C_n^{(\beta,0)}(t;q^2) \tag{8}$$

(which can also be expressed in terms of big q-Jacobi polynomials, as we can see in [15, formulas (4.48) and (4.49)]).

4.3.2. *q-Fourier-Neumann type series.* Now, given $\alpha > -1$, we define the q-Neumann functions by

$$\mathcal{J}_{\alpha,n}(x;q^2) = \frac{J_{\alpha+n+1}(xq^{[\frac{n+1}{2}]};q^2)}{x^{\alpha+1}},$$

where $[\frac{n+1}{2}]$ denotes the biggest integer less or equal than $\frac{n+1}{2}$. The identity

$$\begin{aligned} &\int_0^\infty x^{-\lambda} J_\mu(q^m x;q^2) J_\nu(q^n x;q^2)\, d_qx \\ &= \begin{cases} (1-q)q^{n(\lambda-1)+(m-n)\mu} \dfrac{(q^{1+\lambda+\nu-\mu},q^{2\mu+2};q^2)_\infty}{(q^{1-\lambda+\nu+\mu},q^2;q^2)_\infty} \\ \qquad \times {}_2\phi_1\left(\begin{matrix} q^{1-\lambda+\mu+\nu},q^{1-\lambda+\mu-\nu} \\ q^{2\mu+2} \end{matrix}\,\middle|\, q^2;q^{2m-2n+1+\lambda+\nu-\mu}\right), \\ (1-q)q^{m(\lambda-1)+(n-m)\nu} \dfrac{(q^{1+\lambda+\mu-\nu},q^{2\nu+2};q^2)_\infty}{(q^{1-\lambda+\mu+\nu},q^2;q^2)_\infty} \\ \qquad \times {}_2\phi_1\left(\begin{matrix} q^{1-\lambda+\nu+\mu},q^{1-\lambda+\nu-\mu} \\ q^{2\nu+2} \end{matrix}\,\middle|\, q^2;q^{2n-2m+1+\lambda+\mu-\nu}\right), \end{cases} \end{aligned} \tag{9}$$

was established in [16, 17], and it is valid for $\operatorname{Re}\lambda < \operatorname{Re}(\mu+\nu+1)$, m and n integers. It can be checked, by using Heine's transformation formula

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; z\right) = \frac{(b, az; q)_\infty}{(c, z, q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/b, z \\ az \end{matrix} \,\middle|\, q; b\right), $$

that the expressions given on the right-hand side of (9) are equal; but there are some exceptional cases in the previous identity (these exceptional cases were overlooked in [16] and they can be seen in [17], which is a corrected version of the first paper): the integral is only equal to the first part of the right-hand side when $n-m+(1+\lambda+\mu-\nu)/2$ and $(1-\lambda+\nu-\mu)/2$ are non-positive integers, and it is only equal to the second part when $m-n+(1+\lambda+\nu-\mu)/2$ and $(1-\lambda+\mu-\nu)/2$ are non-positive integers.

From (9) we can state the following lemma.

Lemma 1. *Let $\alpha > -1$. Then*

$$ \int_0^\infty J_{\alpha+2n+1}(q^n x; q^2) J_{\alpha+2m+1}(q^m x; q^2)\, \frac{d_q x}{x} = \frac{1-q}{1-q^{2\alpha+4m+2}}\, \delta_{n,m}, $$

for $n, m = 0, 1, 2, \ldots$.

Proof. It is easy to check from (9) that in the case $q^n = q^m$, $\lambda = 1$, and $\mu = \nu$,

$$ \int_0^\infty (J_\mu(q^m x; q^2))^2\, \frac{d_q x}{x} = \frac{1-q}{1-q^{2\mu}}. $$

For the case $n \neq m$, by setting $\lambda = 1$, $\nu = \alpha+2n+1$ and $\mu = \alpha+2m+1$ in (9), it is clear that

$$ \begin{aligned} \int_0^\infty J_{\alpha+2n+1}(q^n x; q^2) J_{\alpha+2m+1}(q^m x; q^2)\, \frac{d_q x}{x} &= (1-q) q^{(m-n)(\alpha+2n+1)} \\ &\times \frac{(q^{2n-2m+2}, q^{2\alpha+4m+4}; q^2)_\infty}{(q^{2\alpha+2n+2m+2}, q^2; q^2)}\, {}_2\phi_1\left(\begin{matrix} q^{2\alpha+2n+2m+2}, q^{2m-2n} \\ q^{2\alpha+4m+4} \end{matrix} \,\middle|\, q^2; q^2\right). \end{aligned} $$

Then, by using the identity (see [8, formula (II.6)])

$$ {}_2\phi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix} \,\middle|\, q; q\right) = \frac{(c/a; q)_n}{(c; q)_n} a^n $$

we deduce that the integral is null, and thus the proof is complete. □

By using the previous lemma with α and $\alpha+1$, and taking into account that $\mathcal{J}_{\alpha,n}(x;q)$ is even or odd according n is even or odd, respectively, we have that $\{\mathcal{J}_{\alpha,n}(x;q)\}_{n\geq 0}$ is an orthogonal system on $L^2(\mathbb{R}, d\mu_{\alpha,q}(x))$, namely

$$ \int_{-\infty}^\infty \mathcal{J}_{\alpha,n}(x; q^2) \mathcal{J}_{\alpha,m}(x; q^2)\, d\mu_{q,\alpha}(x) = \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} \frac{1}{1-q^{2\alpha+2m+2}}\, \delta_{n,m}, $$

for $n, m = 0, 1, 2, \ldots$.

To find the functions required in the ingredient (iii) of our method, we consider

$$ \mathcal{Q}_n^{(\alpha,\beta)}(t; q^2) = (h_{n,q}^{(\beta,\alpha)})^{-1} \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{2+2\beta}; q^2)_\infty} C_n^{(\beta+1/2,\alpha+1/2)}(t; q^2), \tag{10} $$

$$ \mathcal{P}_n^{(\alpha,\beta)}(t; q^2) = C_n^{(\beta+1/2,\alpha+1/2)}(t; q^2), \tag{11} $$

and use the following lemma.

Lemma 2. *Let $\alpha, \beta > -1$, $\alpha+\beta > -1$, and $k = 0, 1, 2, \dots$. Then*

$$\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,k}(\,\cdot\,;q^2))(t) = \frac{(-i)^k q^{[\frac{k}{2}]\beta}}{1-q^{2\alpha+2\beta+2k+2}} \frac{(q^{2\alpha+2\beta+2};q^2)_\infty}{(q^2;q^2)_\infty} \mathcal{Q}_k^{(\alpha,\beta)}(t;q^2), \tag{12}$$

for $t \in \{\pm q^k\}_{k\in\mathbb{Z}}$, and

$$\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,k}(\,\cdot\,;q^2))(t) = q^{-[\frac{k}{2}]\beta}(-i)^k \frac{(q^{2\alpha+2};q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty} \mathcal{P}_k^{(\alpha,\beta)}(t;q^2), \tag{13}$$

for $t \in \{\pm q^k\}_{k\in\mathbb{Z}} \cap [-1,1]$.

The proof of Lemma 2 is contained in subsection 5.2.

4.4. **Main result.**

Theorem 2. *Let $\alpha, \beta > -1$ and $\alpha+\beta > -1$. Then for each $x \in \{\pm q^k\}_{k\in\mathbb{Z}}$ the following expansion holds in $L^2([-1,1], d\mu_{q,\alpha})$:*

$$\begin{aligned} &E_\alpha(ixt;q^2) \\ &= \frac{(q^2;q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty} \sum_{n=0}^\infty i^n q^{-[\frac{n+1}{2}]\beta}(1-q^{2\alpha+2\beta+2n+2})\mathcal{J}_{\alpha+\beta,n}(x;q^2) C_n^{(\beta+1/2,\alpha+1/2)}(t;q^2). \end{aligned} \tag{14}$$

Moreover, for $f \in PW_{\alpha,q}$, we have the orthogonal expansion

$$f(x) = \sum_{n=0}^\infty a_n(f)(1-q^{2\alpha+2\beta+2n+2})\mathcal{J}_{\alpha+\beta,n}(x;q^2)$$

with

$$a_n(f) = \frac{(q^2;q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty} \int_{\mathbb{R}} f(t)\mathcal{J}_{\alpha+\beta,n}(t;q^2)\, d\mu_{q,\alpha+\beta}(t). \tag{15}$$

Furthermore, the series converges uniformly in compact subsets of $\mathbb{R}$.

Proof. We proceed as in the proof of Theorem 2 in [2] by using the appropriate modifications. In the biorthogonal setup given in section 2, let $\Omega = \mathbb{R}$, $I = [-1,1]$, the space $L^2(I, d\mu) = L^2([-1,1], d\mu_{q,\alpha})$, and the kernel $K(x,t) = E_\alpha(ixt;q^2)$, so $\mathcal{K}$ becomes $\mathcal{F}_{\alpha,q}$, the q-analogue of the Dunkl transform defined in (7) (and $\widetilde{K} = \mathcal{F}_{\alpha,q}^{-1}$). Also, consider the Paley-Wiener space $\mathcal{P} = PW_{\alpha,q}$ of subsection 4.2. Finally, for $N = \mathbb{N}\cup\{0\}$, take the biorthonormal system given by $P_n(t) = \mathcal{P}_n^{(\alpha,\beta)}(t;q^2)$ and $Q_n(t) = \mathcal{Q}_n^{(\alpha,\beta)}(t;q^2)$ as in (11) and (10). The result now follows easily from Theorem 1 and Lemma 2. □

Remark 1. Taking the even parts in the identity (14), we deduced the following expansion for the kernel of the q-Hankel transform:

$$\begin{aligned} \frac{J_\alpha(xt;q^2)}{(xt)^\alpha} &= \frac{(q^{2\alpha+2};q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty} \\ &\times \sum_{n=0}^\infty q^{-n\beta}(1-q^{2\alpha+2\beta+4n+2}) \frac{(q^{2\alpha+2\beta+2};q^2)_n}{(q^{2\alpha+2};q^2)_n} \frac{J_{\alpha+\beta+2n+1}(xq^n;q^2)}{x^{\alpha+\beta+1}} p_n^{(\alpha,\beta)}(t^2;q^2), \end{aligned}$$

valid for α and β that satisfy $\alpha, \beta > -1$ and $\alpha+\beta > -1$. Moreover, the functions belonging to the q-Hankel analogue of the Paley-Wiener space, which is the domain of the sampling theorem in [1], can be spanned by systems of q-Neumann functions.

4.5. **Proof of formula** (2)**.** Setting $\alpha = -\frac{1}{2}$ and replacing β by $\beta - \frac{1}{2}$, in (15) we obtain the expansion for the q-exponential function studied in [19] in terms of the little q-Gegenbauer polynomials defined in (8):

$$e(ixt;q^2) = \frac{(q^2;q^2)_\infty}{(q^{2\beta};q^2)_\infty}\, x^{-\beta} \sum_{n=0}^{\infty} i^n q^{-[\frac{n+1}{2}](\beta-\frac{1}{2})}(1-q^{2\beta+2n}) J_{\beta+n}(xq^{[\frac{n+1}{2}]};q^2) C_n^{\beta}(t;q^2).$$

5. Technical lemmas

In this section we present the calculations which provide us with the q-analogues of the results in [2, section 5].

5.1. **Some integrals involving q-Bessel functions.** We will use the transformation

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, q;z\right) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix}\,\middle|\, q; abz/c\right); \tag{16}$$

this formula appears in [9, formula (12.5.3)] subject to the conditions $|z| < 1$ and $|abz| < |c|$, but these restrictions on z and on the parameters can be eliminated because, by analytic continuation, the identity holds on $\mathbb{C}$ for all parameters as an identity of meromorphic functions (se also [8, p. 117]). We also make repeated use of the obvious identity $(a;q)_\infty = (a;q)_n (aq^n;q)_\infty$.

Lemma 3. *For $\alpha, \beta > -1$ with $\alpha + \beta > -1$, and $n = 0, 1, 2, \dots$, let us define*

$$I_-(\alpha,\beta,n)(t,q) = \frac{t^{-\alpha}}{1-q} \int_0^\infty x^{-\beta} J_\alpha(xt;q^2) J_{\alpha+\beta+2n+1}(q^n x;q^2)\, d_q x$$

and

$$I_+(\alpha,\beta,n)(t,q) = \frac{t^{-\alpha}}{1-q} \int_0^\infty x^{\beta} J_\alpha(xt;q^2) J_{\alpha+\beta+2n+1}(q^n x;q^2)\, d_q x.$$

Then, we have

$$I_-(\alpha,\beta,n)(t,q) = q^{n\beta}\, \frac{(q^{2\beta+2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}\, p_n^{(\alpha,\beta)}(t^2;q^2), \quad t \in \{q^m\}_{m\in\mathbb{Z}}, \tag{17}$$

and

$$I_+(\alpha,\beta,n)(t,q) = q^{-n\beta}\, \frac{(q^{2\alpha+n+2};q^2)_\infty}{(q^{2\alpha+2\beta+2n+2};q^2)_\infty}\, p_n^{(\alpha,\beta)}(t^2;q^2), \quad t \in \{q^m\}_{m\in\mathbb{Z}} \cap (0,1]. \tag{18}$$

Remark 2. Note that $I_-(\alpha,\beta,n)(t,q) = 0$ for $t > 1$. This is due to the factor $(t^2q^2;q^2)_\infty$ involved in the formula.

Remark 3. The identity (17) can be interpreted in terms of the q-Hankel transform in the way

$$H_{\alpha,q}(\mathcal{J}_{\alpha+\beta,2n}(\cdot;q^2))(t) = q^{n\beta}\, \frac{(q^{2\beta+2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}\, p_n^{(\alpha,\beta)}(t^2;q^2)$$

and, as a consequence of the inversion formula, it is also verified that

$$H_{\alpha,q}\left(q^{n\beta}\, \frac{(q^{2\beta+2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty} \frac{((\cdot)^2q^2;q^2)_\infty}{((\cdot)^2q^{2\beta+2};q^2)_\infty}\, p_n^{(\alpha,\beta)}((\cdot)^2;q^2)\right)(t) = \mathcal{J}_{\alpha+\beta,2n}(t;q^2).$$

Proof of Lemma 3. We start evaluating $I_-(\alpha,\beta,n)(t,q)$ for $t \in \{q^m\}_{m\in\mathbb{Z}}$. To this end, we take in (9) $q^m = t$, $\mu = \alpha$, $\nu = \alpha+\beta+2n+1$ and $\lambda = \beta$. For $t \le 1$ or $t > 1$ and β non-integer we can use the first part of the right-hand side of (9) to compute $I_-(\alpha,\beta,n)(t,q)$ because we are not in the exceptional situations. Then, in these cases,

$$\begin{aligned} &I_-(\alpha,\beta,n)(t,q) \\ &\quad = q^{n(\beta-\alpha-1)} \frac{(q^{2\beta+2n+2}, q^{2\alpha+2}; q^2)_\infty}{(q^{2\alpha+2n+2}, q^2; q^2)_\infty} \, {}_2\phi_1 \left(\begin{matrix} q^{2\alpha+2n+2}, q^{-2n-2\beta} \\ q^{2\alpha+2} \end{matrix} \,\middle|\, q^2; t^2 q^{2\beta+2} \right). \end{aligned}$$

Moreover, the previous identity can be extended to all β if $t > 1$ by continuity of the integral $I_-(\alpha,\beta,n)(t,q)$. Now, applying formula (16) and the definition of $p_n^{(\alpha,\beta)}$ in terms of the little q-Jacobi polynomials, we have

$$\begin{aligned} &I_-(\alpha,\beta,n)(t,q) \\ &\qquad = q^{n(\beta-\alpha-1)} \frac{(q^{2\beta+2n+2}, q^{2\alpha+2}; q^2)_\infty}{(q^{2\alpha+2n+2}, q^2; q^2)_\infty} \\ &\qquad\quad \times \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty} \, {}_2\phi_1 \left(\begin{matrix} q^{-2n}, q^{2\alpha+2\beta+2n+2} \\ q^{2\alpha+2} \end{matrix} \,\middle|\, q^2; t^2 q^2 \right) \\ &\qquad = q^{n(\beta-\alpha-1)} \frac{(q^{2\beta+2n+2}, q^{2\alpha+2}; q^2)_\infty}{(q^{2\alpha+2n+2}, q^2; q^2)_\infty} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty} p_n(t^2; q^{2\alpha}; q^{2\beta}; q^2) \\ &\qquad = q^{n\beta} \frac{(q^{2\beta+2n+2}; q^2)_\infty}{(q^{2n+2}; q^2)_\infty} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty} p_n^{(\alpha,\beta)}(t^2; q^2), \end{aligned}$$

and the proof of (17) is completed.

To prove the second part of the lemma, for $t = q^m \in (0,1]$ we evaluate the integral by considering the first part of the right-hand side of (9) and choosing the parameters $\mu = \alpha$, $\nu = \alpha+\beta+2n+1$ and $\lambda = -\beta$. (The first part of the right-hand side of (9) cannot be used for $t > 1$, because we are then in the exceptional case. An expression for $I_+(\alpha,\beta,n)(t,q)$ valid for all t could be obtained with the second part of the right-hand side of (9).) This results in

$$\begin{aligned} I_+(\alpha,\beta,n)(t,q) &= q^{-n(\beta+\alpha+1)} \frac{(q^{2n+2}, q^{2\alpha+2}; q^2)_\infty}{(q^{2\alpha+2\beta+2n+2}, q^2; q^2)_\infty} \\ &\qquad \times {}_2\phi_1 \left(\begin{matrix} q^{-2n}, q^{2\alpha+2\beta+2n+2} \\ q^{2\alpha+2} \end{matrix} \,\middle|\, q^2; t^2 q^2 \right) \\ &= q^{-n(\beta+\alpha+1)} \frac{(q^{2n+2}, q^{2\alpha+2}; q^2)_\infty}{(q^{2\alpha+2\beta+2n+2}, q^2; q^2)_\infty} p_n(t^2; q^{2\alpha}; q^{2\beta}; q^2) \\ &= q^{-n\beta} \frac{(q^{2\alpha+n+2}; q^2)_\infty}{(q^{2\alpha+2\beta+2n+2}; q^2)_\infty} p_n^{(\alpha,\beta)}(t^2; q^2). \end{aligned}$$

In this manner, we have proved (18) and the proof of the lemma is finished. □

5.2. **Proof of Lemma 2.** Let us analyze the case $k = 2n$ for (12). By decomposing on even and odd functions we can write

(19)

$$\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,; q^2))(t) = \frac{1}{1-q} \int_0^\infty \frac{J_{\alpha+\beta+2n+1}(q^n x; q^2)}{x^{\alpha+\beta+1}} \frac{J_\alpha(xt;q^2)}{(xt)^\alpha} x^{2\alpha+1} \, d_q x.$$

Then, for $t > 0$, $\alpha, \beta > -1$, and $\alpha + \beta > -1$, by using (17), it is verified that

$$\begin{aligned}\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) &= q^{n\beta}\,\frac{(q^{2\beta+2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty}\,\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}\,p_n^{(\alpha,\beta)}(t^2;q^2)\\ &= (-1)^n q^{n\beta}\,\frac{(q^{2\alpha+2};q^2)_n}{(q^{2\alpha+2\beta+2};q^2)_n}\,\frac{(q^{2\beta+2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty}\\ &\qquad\times\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\beta+2};q^2)_\infty}C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\\ &= \frac{(-1)^n q^{n\beta}}{1-q^{2\alpha+2\beta+4n+2}}\,\frac{(q^{2\alpha+2\beta+2};q^2)_\infty}{(q^2;q^2)_\infty}\,\mathcal{Q}_{2n}^{(\alpha,\beta)}(t;q^2).\end{aligned}$$

For $t < 0$, let us make in (19) the change $t_1 = -t$, use the evenness of the function $J_\alpha(z)/z^\alpha$, proceed as in the case $t > 0$, and undo the change. Then, for $k = 2n$, we get

$$\mathcal{F}_{\alpha,q}(\mathcal{J}_{\alpha+\beta,k}(\,\cdot\,;q^2))(t) = \frac{(-i)^k q^{\frac{k}{2}\beta}}{1-q^{2\alpha+2\beta+2k+2}}\,\frac{(q^{2\alpha+2\beta+2};q^2)_\infty}{(q^2;q^2)_\infty}\,\mathcal{Q}_k^{(\alpha,\beta)}(t;q^2)\chi_{[-1,1]}(t).$$

The case $k = 2n+1$ works in a similar way.

Now, we are going to prove (13). Again let us analyze the case $k = 2n$. By decomposing on even and odd functions we can write

$$\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) = \frac{1}{1-q}\int_0^\infty x^{2\beta}\,\frac{J_{\alpha+\beta+2n+1}(q^n x;q^2)}{x^{\alpha+\beta+1}}\,\frac{J_\alpha(xt;q^2)}{(xt)^\alpha}x^{2\alpha+1}\,d_qx.$$

Then, if $0 < t < 1$, we can use (18) to obtain

$$\begin{aligned}\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,2n}(\,\cdot\,;q^2))(t) &= q^{-n\beta}\,\frac{(q^{2\alpha+n+2};q^2)_\infty}{(q^{2\alpha+2\beta+2n+2};q^2)_\infty}\,p_n^{(\alpha,\beta)}(x^2;q^2)\\ &= (-1)^n q^{-n\beta}\,\frac{(q^{2\alpha+2};q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty}C_{2n}^{(\beta+1/2,\alpha+1/2)}(t;q^2)\\ &= (-1)^n q^{-n\beta}\,\frac{(q^{2\alpha+2};q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty}\mathcal{P}_{2n}^{(\alpha,\beta)}(t;q^2).\end{aligned}$$

For $t < 0$ we proceed as in the previous identity. Then, for $k = 2n$, we get

$$\mathcal{F}_{\alpha,q}(|\cdot|^{2\beta}\mathcal{J}_{\alpha+\beta,k}(\,\cdot\,;q^2))(t) = q^{-\frac{k}{2}\beta}(-i)^k\,\frac{(q^{2\alpha+2};q^2)_\infty}{(q^{2\alpha+2\beta+2};q^2)_\infty}\mathcal{P}_k^{(\alpha,\beta)}(t;q^2).$$

The case $k = 2n+1$ can be checked with the same arguments.

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